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Nonlinear wave reflection from a submerged circular cylinder

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The paper discusses analytically the nonlinear wave reflection caused by a circular cylinder, submerged under a free surface in water of infinite depth. For monochromatic incident waves it is shown that there is no reflection of order m and frequency $m\omega$ (m integer). This means that the dominant part of the mode of frequency $m\omega$ is not reflected. For bichromatic incident waves it is found that the second-order wave with 'sum frequency' has no reflection. It is shown that the x- and y-components of the oscillatory force of order m and frequency $m\omega$ have identical amplitudes and a phase-difference $\frac{1}{2}\pi$. A corresponding result is also true for bichromatic waves.

1. Introduction

Normally a body submerged under a free surface reflects part of the incident wave motion, except possibly for some special values of the incident frequency. It was shown, however, by Dean (1948) that if the submerged body is a circular cylinder with axis parallel to the crests of the incident wave, and the fluid layer is of infinite depth, the first-order (in wave amplitude) coefficient of reflection is zero for all incident frequencies and all submergences of the body. A complete solution of the linear problem was given by Ursell (1950) who applied a multipole expansion.

Recently Vada (1987) solved numerically the first- and second-order diffraction problem in two dimensions for a submerged cylinder of arbitrary form. He calculated the first- and second-order forces on the body by direct pressure integration, and the second-order reflection and transmission coefficients. In particular, he noted that the magnitude of the second-order reflection coefficient for a submerged circular cylinder was of the same order as the accuracy in his numerical scheme. Very recently Friis (1990), McIver & McIver (1990) and Wu (1991), independently, have shown analytically that the second-order reflection coefficient for the submerged circular cylinder is identically zero. Friis (1990) uses a method applied earlier by Grue & Palm (1985) for a submerged circular cylinder in a uniform current, whereas the two other papers are both based on a formula for the second-order reflection coefficient expressed by the first-order solution only.

In the present paper these results will be generalized. We shall consider the reflection of a Fourier mode with frequency $m\omega$ (the m-harmonic mode) where m is an arbitrary positive number and ω the frequency of the incident wave. This mode is composed of terms of order m, $m+2$, $m+4$, etc. The dominant term is the lowest-order one, i.e. the term of order m. We shall prove that the reflection of this term is identically zero.

We shall also consider incident bichromatic waves (and shortly also multi-chromatic waves) whereby we are able to study second-order effects due to an arbitrary incident wave spectrum. Lately, there has been considerable interest in this problem, mainly due to the fact that a moored, floating body – a ship or oil platform, for instance – may be in resonance with the second-order load, with a frequency of either the difference or the sum of the frequencies of two incident waves. We shall restrict ourselves to consider only sum frequencies and show that the reflection coefficient for the second-order motion is zero. It is discussed below how this result may be generalized to higher-order terms.

Ogilvie (1963) has shown that in the first-order problem the oscillatory forces in the x- and y-directions have identical amplitudes and a phase difference $\frac{1}{2}\pi$. This result will be extended to be valid to second order also for incident bichromatic waves when only sum frequencies are considered. The result will be further extended to any m-harmonic mode of order m.

Section 2 contains the formulation of the problem. In §3 the first-order problem, slightly generalized for later use, is discussed. In §4 we consider the reflection to second and higher order for monochromatic incident waves, and in §5 the reflection for bichromatic incident waves. The oscillatory forces are discussed in §6, and §7 is summary and discussion.

2. Formulation of the problem

We consider first an incident two-dimensional periodic wave with amplitude A and frequency ω which is scattered by a submerged circular cylinder with radius a and contour C, and with axis parallel to the wave crests (cf. figure 1). The fluid is assumed to be incompressible and the motion irrotational. A velocity potential $\hat{\Phi}$ satisfying the Laplace equation then exists.

The boundary conditions on the free surface are

$$\hat{\Phi}_{tt}+g\hat{\Phi}_Y+2\nabla\hat{\Phi}\cdot\nabla\hat{\Phi}_t+\tfrac{1}{2}\nabla\hat{\Phi}\cdot\nabla(\nabla\hat{\Phi}\cdot\nabla\hat{\Phi})=0 \quad (Y=\hat{\eta}), \tag{1}$$

$$g\hat{\eta}=-\hat{\Phi}_t-\tfrac{1}{2}(\nabla\hat{\Phi})^2 \quad (Y=\hat{\eta}), \tag{2}$$

where $\hat{\eta}$ denotes the elevation of the free surface. X and Y are defined in figure 1, and t denotes time. We assume that the fluid depth is infinite and the cylinder restrained. This gives the additional conditions

$$\hat{\Phi}=0 \quad (Y=-\infty), \tag{3}$$

$$\frac{\partial\hat{\Phi}}{\partial n}=0 \quad (X,Y)\in C. \tag{4}$$

Here n denotes the normal derivative, chosen positive out of the fluid. We introduce the dimensionless quantities

$$\left.\begin{aligned} x=\frac{X}{a},\quad y=\frac{Y}{a},\quad \epsilon=\frac{A}{a},\quad h=\frac{H}{a},\quad K=\frac{a\omega^2}{g},\quad \tau=\omega t,\quad \eta(x,\tau)=\frac{\hat{\eta}(X,t)}{a},\\ \Phi(x,y,\tau)=\frac{\hat{\Phi}(X,Y,t)}{\omega a^2},\quad p(x,y,\tau)=\frac{\hat{p}(X,Y,t)}{\rho g a},\quad F(\tau)=\frac{\hat{F}(t)}{\rho g a^2},\end{aligned}\right\} \tag{5}$$

where $\hat{p}$ is pressure, ρ is density and $\hat{F}$ is the force per unit length of the cylinder. It is assumed that ϵ is small and that the potential can be expanded in a series

$$\Phi=\sum_1^\infty \epsilon^n\,\Phi_n. \tag{6}$$

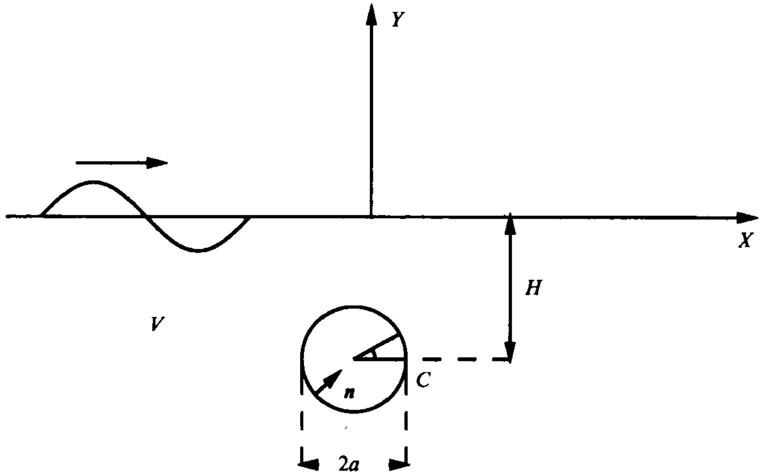

FIGURE 1. Definition sketch.

ϵ is proportional to the Keulegan–Carpenter number $(A/a)\pi\exp(-Kh)$. We also require that the wave steepness ϵK is small, which restricts the magnitude of K to be $O(1)$. The motion is considered to be periodic in time so we may write

$$\Phi_1 = \mathrm{Re}\{(\phi_0(x,y)+\phi_\mathrm{D}(x,y))\,\mathrm{e}^{\mathrm{i}\tau}\}, \tag{7}$$

$$\Phi_2 = \mathrm{Re}\{\phi_{20}(x,y)+\phi_{22}(x,y)\,\mathrm{e}^{2\mathrm{i}\tau}\}, \tag{8}$$

$$\Phi_m = \mathrm{Re}\{\phi_{m0}(x,y)+\phi_{m1}(x,y)\,\mathrm{e}^{\mathrm{i}\tau}+\ldots\phi_{mm}(x,y)\,\mathrm{e}^{\mathrm{i}m\tau}\}, \tag{9}$$

where i is the imaginary unit and m an arbitrary integer. ϕ_0 is the velocity potential for the incident wave (with unit amplitude), given by

$$\phi_0(x,y) = \frac{\mathrm{i}}{K}\mathrm{e}^{Ky-\mathrm{i}Kx} = \frac{\mathrm{i}}{K}\mathrm{e}^{-\mathrm{i}Kz}, \tag{10}$$

where
$$z = x+\mathrm{i}y. \tag{11}$$

The boundary conditions for $\phi_0, \phi_\mathrm{D}, \ldots, \phi_{mm}$ are obtained by introducing (7)–(9) in (1) and (2) and developing (1) and (2) in a Taylor series around $y = 0$, eliminating η. For ϕ_D and ϕ_{mm} we obtain the following boundary conditions:

$$(\phi_\mathrm{D})_y - K\phi_\mathrm{D} = 0 \quad (y=0), \tag{12}$$

$$\phi_\mathrm{D} = 0 \quad (y=-\infty), \tag{13}$$

$$(\phi_{mm})_y - m^2K\phi_{mm} = f_m(x) \quad (y=0), \tag{14}$$

$$\phi_{mm} = 0 \quad (y=-\infty). \tag{15}$$

Introducing ϕ_1 defined by
$$\phi_1 = \phi_0+\phi_\mathrm{D}, \tag{16}$$

we also have
$$(\phi_1)_y - K\phi_1 = 0 \quad (y=0), \tag{17}$$

$$\phi_1 = 0 \quad (y=-\infty), \tag{18}$$

$$\frac{\partial\phi_1}{\partial n} = 0, \quad \frac{\partial\phi_{mm}}{\partial n} = 0 \quad (x,y)\in C. \tag{19}$$

Here we only give $f_m(x)$ for $m = 2$,

$$\begin{aligned} f_2(x) &= \tfrac{1}{2}\mathrm{i}K\left[-2\nabla\phi_1\cdot\nabla\phi_1+\phi_1\frac{\partial}{\partial y}((\phi_1)_y-K\phi_1)\right]_{y=0} \\ &= -\tfrac{1}{2}\mathrm{i}K[3K^2(\phi_1)^2+(\phi_1)_{xx}\phi_1+2(\phi_1)_x^2]_{y=0}. \end{aligned} \tag{20}$$

In addition the potentials must satisfy the Laplace equation and the radiation conditions at $x = \pm\infty$. The radiation conditions for ϕ_D are

$$(\phi_D)_x \pm iK\phi_D = 0 \quad (x = \pm\infty), \tag{21}$$

stating that the motion at infinity is an outgoing wave. From (21) and (20) we get

$$\lim_{x\to\infty} f_2(x) = 0, \quad \lim_{x\to-\infty} f_2(x) = 4iKR_1, \tag{22}$$

where R_1 is the first-order reflection coefficient, and from (14) (with $m = 2$) and (22), assuming outgoing waves at infinity (Vada 1987),

$$(\phi_{22})_x + 4iK\phi_{22} = 0 \quad (x = \infty), \tag{23}$$

$$(\phi_{22})_x - 4iK\phi_{22} = -4KR_1 \quad (x = -\infty). \tag{24}$$

We note that for $R_1 = 0$, ϕ_{22} at $x = \pm\infty$ describes a free, outgoing wave.

The time-dependence of the pressure, the force and the free-surface elevation is separated out in the same way as for the velocity potential. From the Bernoulli equation we obtain

$$p_1(x, y) = -Ki\phi_1, \tag{25}$$

$$p_{22}(x, y) = -2Ki\phi_{22} - \tfrac{1}{4}K(\nabla\phi_1)^2, \tag{26}$$

where p_1 is the first-order pressure and p_{22} the oscillatory part of the second-order pressure.

The surface elevation is found from the dynamic boundary condition (2) to be

$$\eta_1(x) = -iK\phi_1(x, 0), \tag{27}$$

$$\eta_{22}(x) = -K(2i\phi_{22} + \tfrac{3}{4}K^2(\phi_1)^2 + \tfrac{1}{4}(\phi_1)_x^2)|_{y=0}. \tag{28}$$

Introducing in (28) T_1, T_2 and R_2, the first- and second-order transmission coefficients and the second-order reflection coefficient, respectively, we find (Vada 1987)

$$\eta_{22}(x) = T_2\,e^{-i4Kx} - \tfrac{1}{2}KT_1^2\,e^{-i2Kx} \quad (x = \infty), \tag{29}$$

$$\eta_{22}(x) = R_2\,e^{i4Kx} - \tfrac{1}{2}K\,e^{-i2Kx} - \tfrac{1}{2}R_1^2K\,e^{i2Kx} \quad (x = -\infty). \tag{30}$$

3. The first-order problem

We consider in this section the first-order problem with an incoming wave which, for later reference, has a somewhat generalized form. The Green function for an oscillating source fulfilling the Laplace equation, the boundary conditions at $y = 0$ and $y = -\infty$ and the radiation conditions is given by Wehausen & Laitone (1960) as

$$G^*(x, y, x', y', K, \tau) = \mathrm{Re}\,(G(x, y, x', y', K)\,e^{i\omega\tau}), \tag{31}$$

where

$$G(x, y, x', y', K) = \log r/r^* + G'(x, y, x', y', K) \tag{32}$$

with

$$G' = -2\int_0^\infty \frac{e^{k(y-y')}\cos k(x-x')}{k-K}\,dk = -\int_0^\infty \frac{e^{ik(\bar{z}-z')}}{k-K}\,dk - \int_0^\infty \frac{e^{-ik(z-\bar{z}')}}{k-K}\,dk. \tag{33}$$

The contour of integration is deformed above the pole in the complex k-plane. (x', y') denote the source coordinates and (x, y) the space coordinates. r and r^* are given by

$$r = ((x-x')^2 + (y-y')^2)^{\frac{1}{2}}, \tag{34}$$

$$r^* = ((x-x')^2 + (y+y')^2)^{\frac{1}{2}}, \tag{35}$$

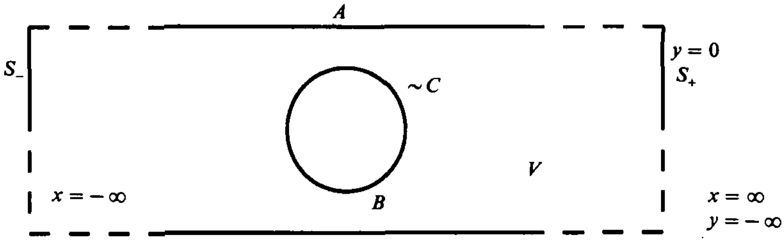

FIGURE 2. Path of integration.

and a bar denotes complex conjugate; z is defined by (11). The asymptotic values of (32) are found by contour integration to be

$$G = 2\mathrm{i}\pi\, \mathrm{e}^{-\mathrm{i}Kz+\mathrm{i}K\bar{z}'} \quad (x = \infty), \tag{36}$$

$$G = 2\mathrm{i}\pi\, \mathrm{e}^{\mathrm{i}K\bar{z}-\mathrm{i}Kz'} \quad (x = -\infty). \tag{37}$$

We obtain a formula for ϕ_1 by applying Green's theorem for the Green function (32) and ϕ_1 on the closed figure indicated in figure 2. Using

$$\frac{\partial \phi_1}{\partial x} = \mathrm{i}K\phi_1 - 2\mathrm{i}K\phi_0 \quad (x = -\infty), \tag{38}$$

$$\frac{\partial \phi_1}{\partial x} = -\mathrm{i}K\phi_1 \quad (x = \infty) \tag{39}$$

together with (17), (18) and (19), and evaluating the integral at $x = -\infty$ by applying the asymptotic value (37) of the Green function, we obtain

$$-\int_C \phi_1 \frac{\partial G}{\partial n'} \mathrm{d}s' + \pi\phi_1 = 2\pi\phi_0 \quad ((x,y) \in C), \tag{40}$$

$$\int_C \phi_1 \frac{\partial G}{\partial n'} \mathrm{d}s' = 2\pi(\phi_1 - \phi_0) = 2\pi\phi_{\mathrm{D}} \quad ((x,y) \in V), \tag{41}$$

where $\partial/\partial n'$ denotes the normal derivatives out of the fluid domain.

At the contour C we have

$$z = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}h, \quad \partial z/\partial n = -\mathrm{e}^{\mathrm{i}\theta} \quad (z = C). \tag{42}$$

Hence from (36) and (37)

$$\frac{\partial G}{\partial n'} = -2\pi K\, \mathrm{e}^{\mathrm{i}K\bar{z}-\mathrm{i}Kz'}\, \mathrm{e}^{\mathrm{i}\theta'} \quad (z' \in C) \quad (x = -\infty), \tag{43}$$

$$\frac{\partial G}{\partial n'} = 2\pi K\, \mathrm{e}^{-\mathrm{i}Kz+\mathrm{i}K\bar{z}'}\, \mathrm{e}^{-\mathrm{i}\theta'} \quad (z' \in C) \quad (x = \infty). \tag{44}$$

Equation (43) introduced in (41) then gives

$$\phi_{\mathrm{D}} = -K\, \mathrm{e}^{\mathrm{i}K\bar{z}} \int_0^{2\pi} \phi_1\, \mathrm{e}^{-\mathrm{i}Kz'+\mathrm{i}\theta'}\, \mathrm{d}\theta' \quad (z' \in C) \quad (x = -\infty). \tag{45}$$

We assume that $\phi_1(\theta')$ may be developed in a Fourier series

$$\phi_1(\theta') = \sum_0^\infty A_{1m}\, \mathrm{e}^{\mathrm{i}m\theta'} + \sum_1^\infty B_{1m}\, \mathrm{e}^{-\mathrm{i}m\theta'}, \tag{46}$$

where A_{1m} and B_{1m} are complex constants.

Developing $\exp(-iKz')$ in a power series and using (42), we note that in the integral (45) only B_{1m}-terms can give contributions. We shall show that $B_{1m} = 0$, which gives $R_1 = 0$.

For later use we generalize (40), writing

$$-\int_C \phi_1 \frac{\partial G}{\partial n'} \mathrm{d}\theta' + \pi\phi_1 = f(\mathrm{e}^{\mathrm{i}\theta}) \; ((x,y) \in C), \tag{47}$$

where $f(x)$ must have an expansion in powers x^m, where $m \geqslant 0$ is integer. Obviously (40) is a special case of (47). We Fourier transform the integral equation by multiplying the equation with $(1/2\pi)\exp(im\theta)$, where m is positive, and integrating from 0 to 2π. The Fourier transform of the right-hand side of (47) is zero. The last term on the left has the Fourier transform

$$\pi B_{1m}. \tag{48}$$

We split $\partial G/\partial n'$ according to (32) and consider first $\partial/\partial n'(\log r)$, which may be written

$$\frac{\partial}{\partial n'}\log r = \frac{1}{r}\frac{\partial r}{\partial n'} = -\frac{1}{r}\cos(r,n') = -\tfrac{1}{2}. \tag{49}$$

Hence this term makes no contribution to the Fourier transform of (47).

The next term may be written

$$-\frac{\partial}{\partial n'}\log r^* = \mathrm{Re}\frac{1}{z-\bar{z}'}\frac{\partial \bar{z}'}{\partial n'} = \frac{1}{2}\frac{\mathrm{e}^{-\mathrm{i}\theta'}}{2\mathrm{i}h+\mathrm{e}^{-\mathrm{i}\theta'}-\mathrm{e}^{\mathrm{i}\theta}} - \frac{1}{2}\frac{\mathrm{e}^{\mathrm{i}\theta'}}{2\mathrm{i}h+\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'}}. \tag{50}$$

Developing the last term in a power series, we find

$$-\frac{1}{2}\frac{\mathrm{e}^{\mathrm{i}\theta'}}{2\mathrm{i}h+\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'}} = -\frac{\mathrm{e}^{\mathrm{i}\theta'}}{4\mathrm{i}h}\sum_0^\infty \left(\frac{\mathrm{i}}{2h}\right)^n (\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})^n. \tag{51}$$

We notice that this sum contains terms proportional to $\mathrm{e}^{-\mathrm{i}n\theta}$ and therefore makes a contribution to the Fourier transform. The coefficients are of the form $\mathrm{e}^{\mathrm{i}n\theta'}$ (not $\mathrm{e}^{-\mathrm{i}n\theta'}$). Hence this contribution leads only to B_{1m} – terms and no A_{1m}-terms, and will be of the form

$$\sum_1^\infty \beta_{nm} B_{1n} \quad (m = 1, \infty). \tag{52}$$

Similarly it is seen that the first term in (50) makes no contribution to the Fourier transform.

We next consider G' and focus on the first integral on the right-hand side of (33). Developing the exponential function in a power series, we obtain

$$-\frac{\partial}{\partial n'}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}k(\bar{z}-z')}}{k-K}\mathrm{d}k = -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta'}\int_0^\infty \frac{k\,\mathrm{e}^{-2kh}}{k-K}\sum_{n=0}^\infty \frac{1}{n!}(\mathrm{i}k)^n(\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta'})^n\,\mathrm{d}k. \tag{53}$$

Exactly as in the previous discussion this sum leads to a contribution of the form (52) whereas the remaining part of (33) gives no contribution.

Collecting the terms obtained by the Fourier transform of (47), we have

$$\pi B_{1m} + \sum_1^\infty \beta_{nm} B_{1n} = 0 \quad (m = 1, \infty), \tag{54}$$

where β_{nm} now is the sum of the contributions from (50) and (53). Assuming that the infinite determinant is non-vanishing (there are no irregular frequencies here since the body is fully submerged) and that the series (52) is converging sufficiently strongly, it follows that $B_{1m} = 0$ and thereby $R_1 = 0$.

We thus have

$$\phi_1(\theta) = \sum_0^\infty A_{1n}\,\mathrm{e}^{\mathrm{i}n\theta}. \tag{55}$$

Utilizing (55) and the asymptotic values (43) and (44) for $\partial G/\partial n'$, we find from (41) and (10)

$$\phi_1 = \phi_0 + \phi_\mathrm{D} = -\mathrm{i}\left(-\frac{1}{K}+h_1(K)\right)\mathrm{e}^{-\mathrm{i}Kx} \quad (x=\infty), \tag{56}$$

$$\phi_1 = \phi_0 = \frac{\mathrm{i}}{K}\mathrm{e}^{-\mathrm{i}Kx} \quad (x=-\infty), \tag{57}$$

where
$$h_1(K) = 2\pi\,\mathrm{e}^{-Kh}\sum_1^\infty A_{1m}\frac{(K\mathrm{i})^m}{(m-1)!}. \tag{58}$$

For later use we define
$$h_J(K) = 2\pi\,\mathrm{e}^{-Kh}\sum_1^\infty A_{Jm}\frac{(K\mathrm{i})^m}{(m-1)!}, \tag{59}$$

$$g_J(K) = 2\pi\,\mathrm{e}^{-J^2Kh}\sum_1^\infty A_{Jm}\frac{(J^2K\mathrm{i})^m}{(m-1)!}. \tag{60}$$

We also need ϕ_1 at $y=0$. At the free surface $\partial/\partial n'(\log r/r^*) = 0$ and G reduces to G'. Introducing in (41) the formula (55) for ϕ_1, we notice that only the last term in (33) contributes to the integral. Changing the order of integration, we obtain

$$\phi_1(x,0) = \frac{\mathrm{i}}{K}\mathrm{e}^{-\mathrm{i}Kx} + \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{i}kx}}{k-K}h_1(k)\,\mathrm{d}k. \tag{61}$$

4. The second- and higher-order waves

Let us first consider the second-order wave ϕ_{22}. This potential fulfils the Laplace equation and the boundary conditions (14), (15) and (19).

ϕ_{22} is the complete second-order velocity potential with frequency 2ω and contains in principle also the incident second-order (Stokes) wave. The velocity potential for the latter vanishes, however, since the fluid depth is infinite. The radiation conditions for ϕ_{22} are given by (23) and (24) with $R_1 = 0$, expressing that at $|x| = \infty$ the waves are free waves, travelling outwards.

As in §3 we apply Green's theorem. Now we choose as the Green function G_2, defined by

$$G_2(x,y,x',y') = G(x,y,x',y',4K), \tag{62}$$

where G is defined by (32) and (33).

Using the facts that ϕ_{22} and G_2 satisfy the same radiation conditions, both are vanishing at $y=-\infty$ and ϕ_{22} fulfils (19), we obtain

$$\int_C \phi_{22}\frac{\partial G_2}{\partial n'}\mathrm{d}s' - \int_{-\infty}^{\infty} G_2(x,y,x',0)f_2(x')\,\mathrm{d}x' = \pi\phi_{22} \quad ((x,y)\in C), \tag{63}$$

$$\int_C \phi_{22}\frac{\partial G_2}{\partial n'}\mathrm{d}s' - \int_{-\infty}^{\infty} G_2(x,y,x',0)f_2(x')\,\mathrm{d}x' = 2\pi\phi_{22} \quad ((x,y)\in V). \tag{64}$$

The second integral in these equations may be simplified considerably. First, in the formula for G_2, $\log r/r^*$ is zero. Furthermore, according to (61) ϕ_1 at $y=0$ is only a function of x through terms of the form $\exp(-ik_1 x')$ where k_1 is positive. Hence $f_2(x)$ is a function of x only through terms of the form $\exp(-i(k_1+k_2)x)$ where k_1 and k_2 are positive. It then follows (see the Appendix) that only the last integral in (33) makes a non-vanishing contribution. Changing the order of integration we find that (63) and (64) may be written

$$-\int_C \phi_{22}\frac{\partial G_2}{\partial n'}\mathrm{d}s' + \pi\phi_{22} = \int_0^\infty \frac{\tilde{f}_2(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k-4K}\mathrm{d}k \quad ((x,y)\in C), \tag{65}$$

$$-\int_C \phi_{22}\frac{\partial G_2}{\partial n'}\mathrm{d}s' + 2\pi\phi_{22} = \int_0^\infty \frac{\tilde{f}_2(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k-4K}\mathrm{d}k \quad ((x,y)\in V), \tag{66}$$

where $\tilde{f}_2(k)$ is the Fourier transform of $f_2(x)$ defined by

$$\tilde{f}_2(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \tag{67}$$

For k negative, $\tilde{f}_2(k) = 0$. Introducing in (65) the formula (42) for z, we note that (65) is of the same type as (47). Hence $\phi_{22}(\theta')$ may be written

$$\phi_{22}(\theta') = \sum_0^\infty A_{2n}\,\mathrm{e}^{\mathrm{i}n\theta'}. \tag{68}$$

The asymptotic value of ϕ_{22} is obtained from (66) by applying contour integration on the last integral and using (68) in the first integral. Developing the exponential function in a power series, we obtain

$$\phi_{22} = -\mathrm{i}(\tilde{f}_2(4K) + g_2(4K))\,\mathrm{e}^{-\mathrm{i}4Kz} \quad (x=\infty), \tag{69}$$

$$\phi_{22} = 0 \quad (\xi = -\infty), \tag{70}$$

where g_2 is defined by (60). Hence the second-order reflection coefficient R_2 is zero.

For later reference we need a formula for ϕ_{22} at $y=0$. We introduce (68) into (66) and note that G_2 may be replaced by G_2' since $\log r/r^* = 0$, and that only the last integral in (33) gives a non-vanishing contribution. Developing the exponential function in a power series, we obtain

$$\phi_{22}(x,0) = \frac{1}{2\pi}\int_0^\infty \frac{\tilde{f}_2(k) + h_2(k)}{k-4K}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{71}$$

We then consider the third-order velocity potential with frequency 3ω, ϕ_{33}, and derive first the radiation conditions. The boundary condition at $y=0$ is, according to (14),

$$(\phi_{33})_y - 9K\phi_{33} = f_3(x), \tag{72}$$

where $f_3(x)$ is obtained by developing (1) and (2) in Taylor series around $y=0$. We notice that $f_3(x)$ consists of terms of the form (i) $\nabla\Phi\cdot\nabla\Phi_t$ and $(\partial/\partial y)(\nabla\Phi\cdot\nabla\Phi_t)\,\eta$, (ii) $\nabla\Phi\cdot\nabla(\nabla\Phi\cdot\nabla\Phi)$, (iii) $(\partial/\partial y)(\Phi_{tt}+g\Phi_y)\,\eta$ and $(\partial^2/\partial y^2)(\Phi_{tt}+g\Phi_y)\,\eta^2$. Here Φ is Φ_1 or the time-dependent part of Φ_2. The terms (i) lead to terms of the form $\nabla\phi_1\cdot\nabla\phi_{22}$ and $(\partial/\partial y)(\nabla\phi_1\cdot\nabla\phi_1)\,\eta_1$. At $|x| = \infty$ ϕ_1 and ϕ_{22} are zero or a function of z only (see (56), (57) and (69), (70)). For arbitrary analytical functions $F(z)$ and $G(z)$, $\nabla F(z)\cdot\nabla G(z)$ is identically zero. Hence at $|x| = \infty$ the terms (i) vanish. For the same reason the terms (ii) are also zero at infinity. Since ϕ_1 and ϕ_{22} at $|x| = \infty$ are free waves in a fluid of infinite depth, $\Phi_{tt}+g\Phi_y$ and its y-derivatives are zero. Hence also the contribution from the terms (iii) vanish.

We thus end up at $|x| = \infty$ with

$$(\phi_{33})_y - 9K\phi_{33} = 0 \quad (y = 0), \tag{73}$$

which shows that the ϕ_{33}-waves are also free waves at infinity. Since the waves are generated at finite values of x, they must travel outwards and the radiation conditions are

$$(\phi_{33})_x \pm \mathrm{i}9K\phi_{33} = 0 \quad (x = \pm\infty). \tag{74}$$

The physical reason for (73) being a homogeneous equation is that ϕ_1 and ϕ_{22} are not reflected and, furthermore, that an incoming (Stokes) wave has no third-order potential with frequency 3ω in a fluid layer of infinite depth.

The procedure is now analogous to that applied for ϕ_{22}. We define the Green function

$$G_3(x, y, x', y') = G(x, y, x', y', 9K), \tag{75}$$

where G is defined by (32). Since ϕ_{33} and G_3 satisfy the same radiation conditions, both are vanishing at $y = -\infty$ and ϕ_{33} fulfils (19), we obtain

$$\int_C \phi_{33} \frac{\partial G_3}{\partial n'} \mathrm{d}s' - \int_{-\infty}^{\infty} G_3(x, y, x', 0) f_3(x') \,\mathrm{d}x' = \pi\phi_{33} \quad ((x, y) \in C), \tag{76}$$

$$\int_C \phi_{33} \frac{\partial G_3}{\partial n'} \mathrm{d}s' - \int_{-\infty}^{\infty} G_3(x, y, x', 0) f_3(x') \,\mathrm{d}x' = 2\pi\phi_{33} \quad ((x, y) \in V). \tag{77}$$

From (71) we see that $\phi_{22}(x)$ at $y = 0$ is a function of x only through terms of the form $\exp(-\mathrm{i}k_1 x)$ where k_1 is positive. Since $\phi_1(x)$ is also a function of x of the same form, it follows, exactly as for ϕ_{22}, that (76) and (77) may be written (see Appendix)

$$-\int_C \phi_{33} \frac{\partial G_3}{\partial n'} \mathrm{d}s' + \pi\phi_{33} = \int_0^{\infty} \frac{\tilde{f}_3(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k - 9K} \mathrm{d}k \quad ((x, y) \in C), \tag{78}$$

$$-\int^C \phi_{33} \frac{\partial G_3}{\partial n'} \mathrm{d}s' + 2\pi\phi_{33} = \int_0^{\infty} \frac{\tilde{f}_3(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k - 9K} \mathrm{d}k \quad ((x, y) \in V), \tag{79}$$

where $\tilde{f}_3$ is the Fourier transform of f_3. Introducing in (78) the formula (42) for z, we see that (78) is of the same type as (47). Hence ϕ_{33} takes the form

$$\phi_{33}(\theta') = \sum_0^{\infty} A_{3n}\,\mathrm{e}^{\mathrm{i}n\theta'}. \tag{80}$$

The asymptotic form of ϕ_{33} is found in exactly the same way as for ϕ_{22}. We obtain

$$\phi_{33} = -\mathrm{i}(\tilde{f}_3(9K) + g_3(9K))\,\mathrm{e}^{-\mathrm{i}9Kz} \quad (x = \infty), \tag{81}$$

$$\phi_{33} = 0 \quad (x = -\infty). \tag{82}$$

Hence the third-harmonic mode of third order is not reflected.

Furthermore, ϕ_{33} at $y = 0$ is obtained in the same way as we derived formula (71) for ϕ_{22}. We find

$$\phi_{33}(x, 0) = \frac{1}{2\pi} \int_0^{\infty} \frac{\tilde{f}_3(k) + h_3(k)}{k - 9K} \mathrm{e}^{-\mathrm{i}kx} \,\mathrm{d}k. \tag{83}$$

We may now prove that the amplitude of the velocity potential of order m and frequency $m\omega, \phi_{mm}$, is zero at $x = -\infty$. The boundary condition at $y = 0$ is given by (14). $f_m(x)$ consists of products of various ϕ_{nn} (and ϕ_1) where $n < m$. Let us assume for the moment that ϕ_{nn} for $x = \infty$ is of the form of a constant times $\exp(-\mathrm{i}n^2Kz)$, for all n, and zero for $x = -\infty$. As in the discussion for $n = 3$, we divide $f_m(x)$ in three groups where groups (i) and (ii) consist of terms proportional to $\nabla\Phi \cdot \nabla\Phi$ (or $\nabla\Phi \cdot \nabla\Phi_t$) and its y-derivatives. Since at $|x| = \infty$ all ϕ_{nn} (and ϕ_1) are functions of z only, or zero, the contributions from these groups vanish. Furthermore, since ϕ_{nn} is assumed to be free waves at infinity and the fluid layer is of infinite depth, $\phi_{tt} + g\Phi_y$ and all its y-derivatives are zero at $y = 0$. Hence we obtain

$$(\phi_{mm})_y - m^2K\phi_{mm} = 0 \quad (y = 0, |x| = \infty). \tag{84}$$

Similarly as for $m = 3$, we conclude that the radiation condition for ϕ_{mm} is

$$(\phi_{mm})_x \pm \mathrm{i}m^2K\phi_{mm} = 0 \quad (x = \pm\infty). \tag{85}$$

The physical reason for (84) being a homogeneous equation is that all ϕ_{nn} (and ϕ_1) have no reflection and that the incoming (Stokes) wave has no mode in the velocity potential of order m and frequency $m\omega$.

Using now the Green function

$$G_m(x, y, x', y') = G(x, y, x', y', m^2K) \tag{86}$$

we obtain formulae identical to (76) and (77) except that subscript 3 is replaced by m. As a second assumption we now assume that all $\phi_{nn}(x, 0), n < m$, are a function of x only through terms of the form $\exp(-\mathrm{i}kx)$ where k is positive. We then obtain, as for $m = 3$,

$$-\int_C \phi_{mm} \frac{\partial G_m}{\partial n'} \mathrm{d}s' + \pi\phi_{mm} = \int_0^\infty \frac{\tilde{f}_m(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k - m^2K} \mathrm{d}k \quad ((x, y) \in C), \tag{87}$$

$$-\int_C \phi_{mm} \frac{\partial G_m}{\partial n'} \mathrm{d}s' + 2\pi\phi_{mm} = \int_0^\infty \frac{\tilde{f}_m(k)\,\mathrm{e}^{-\mathrm{i}kz}}{k - m^2K} \mathrm{d}k \quad ((x, y) \in V). \tag{88}$$

The integral equation (87) is of the same type as (47) and we conclude that ϕ_{mm} takes the form

$$\phi_{mm}(\theta') = \sum_0^\infty A_{mn}\,\mathrm{e}^{\mathrm{i}n\theta'}. \tag{89}$$

From (89) and the asymptotic expressions for G_m we obtain that the asymptotic forms of ϕ_{mm} may be written

$$\phi_{mm} = -\mathrm{i}(\tilde{f}_m(m^2K) + g_m(m^2K))\,\mathrm{e}^{-\mathrm{i}m^2Kz} \quad (x = \infty), \tag{90}$$

$$\phi_{mm} = 0 \quad (x = -\infty). \tag{91}$$

Furthermore, at $y = 0, \phi_{mm}$ is given by

$$\phi_{mm}(x, 0) = \frac{1}{2\pi} \int_0^\infty \frac{\tilde{f}_m(k) + h_m(k)}{k - m^2K} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{92}$$

The two assumptions we have made for ϕ_{nn}, are true for n less than 4. Hence the results (89)–(92) are valid for $m = 4$. Then they must also be true for $m = 5$, etc. and we conclude that they are valid for arbitrary m.

The corresponding surface elevation η_{mm} at $x = -\infty$ is obtained by expanding (2) in a power series about $y = 0$. η_{mm} consist of products of $\phi_1, \phi_{22}, \phi_{33}, \ldots, \phi_{mm}$ and its derivatives. All these quantities are zero at $x = -\infty$, except ϕ_1 which is equal to ϕ_0. Hence at $x = -\infty$ η_{mm} has the value given by the incoming wave. Thus there is no reflection of the surface elevation of the m-harmonic mode of order m.

5. Reflection of incident bichromatic waves

In recent years considerable interest has been shown in studying incident bichromatic waves. The main reason is that an oscillating system, for instance a moored floating body, may perform oscillations with periods which are outside the linear wave spectrum. The second-order forces, however, include forces with 'sum-frequency' and 'difference-frequency', and these may very well be in resonance with the oscillating system.

We consider two incident waves with dimensionless amplitudes A_1, A_2 and dimensionless wavenumbers K_1, K_2, respectively. The velocity potentials of the incident waves are then given by

$$\phi_0^J = \frac{A_J \mathrm{i}}{K_J} \mathrm{e}^{-\mathrm{i}zK_J} \quad (J = 1, 2). \tag{93}$$

Furthermore, let $\phi_1^{(1)}$ and $\phi_1^{(2)}$ denote the velocity potentials for the total first-order waves. We know that the first-order reflection coefficients for the two waves are zero. We shall now show that the second-order reflection coefficients due to the sum-frequencies also vanish. Let ϕ_2^+ denote the corresponding velocity potential. The boundary condition at $y = 0$ may be written

$$(\phi_2^+)_y - K^+\phi_2^+ = f^+(x), \tag{94}$$

where

$$K^+ = \frac{(\omega_1 + \omega_2)^2 a}{g}, \tag{95}$$

with ω_1 and ω_2 denoting the frequencies of the two incident waves. $f^+(x)$ is obtained from (1) and (2) and consists of products of $\phi_1^{(1)}$ and $\phi_1^{(2)}$. Utilizing the asymptotic formulae (56) and (57) for $\phi_1^{(1)}$ and $\phi_1^{(2)}$ we obtain that $f^+(x) = 0$ for $|x| = \infty$. This is because $R_1 = 0$ and the second-order velocity potential for the incoming wave with sum-frequencies is zero. As in §4 we conclude that the radiation condition for ϕ_2^+ is

$$(\phi_2^+)_x \pm iK^+\phi_2^+ = 0 \quad (x = \pm\infty). \tag{96}$$

Following the procedure in §4 we use Green's theorem, choosing as Green function

$$G_2^+(x, y, x', y') = G(x, y, x', y', K^+), \tag{97}$$

where G is defined by (32) and (33). Since ϕ_{22} and G_2 satisfy the same radiation conditions, both are vanishing at $y = -\infty$ and $\partial/\partial n(\phi_2^+) = 0$ at C, we obtain two equations similar to (63) and (64) for ϕ_2^+. $\mathrm{f}^+(x)$ may be written as products of $\phi_1^{(1)}$ and $\phi_1^{(2)}$ and their x-derivatives. With the same arguments as applied for ϕ_{22}, it follows that the two equations for ϕ_2^+ may be written

$$-\int_C \phi_2^+ \frac{\partial G_2^+}{\partial n'} \mathrm{d}s' + \pi\phi_2^+ = \int_0^\infty \frac{\tilde{f}^+(k)\,\mathrm{e}^{\mathrm{i}kz}}{k-K^+} \mathrm{d}k \quad ((x,y) \in C), \tag{98}$$

$$-\int_C \phi_2^+ \frac{\partial G_2^+}{\partial n'} \mathrm{d}s' + 2\pi\phi_2^+ = \int_0^\infty \frac{\tilde{f}^+(k)\,\mathrm{e}^{\mathrm{i}kz}}{k-K^+} \mathrm{d}k \quad ((x,y) \in V), \tag{99}$$

where $\tilde{f}^+(k)$ is the Fourier transform of $f^+(x)$. Introducing (42) we see that (98) is of the same form as (47). Hence ϕ_2^+ is of the form

$$\phi_2^+(\theta') = \sum_0^\infty A_{2n}^+ \,\mathrm{e}^{\mathrm{i}n\theta'}. \tag{100}$$

Introducing (100) in (99) we obtain

$$\phi_2^+ = -\mathrm{i}(\tilde{f}^+(K^+) + h_2^+(K^+))\,\mathrm{e}^{-\mathrm{i}K^+z} \quad (x = +\infty), \tag{101}$$

$$\phi_2^+ = 0 \quad (x = -\infty), \tag{102}$$

where h_2^+ is defined by (59), replacing A_{2m} with A_{2m}^+. Hence we have shown that the reflection coefficient for ϕ_2^+ is zero.

We can now proceed as in §4 and show that $\phi_2^+(x,0)$ has a form similar to (71). Considering then three incident waves, with velocity potential ϕ_0^J given by (93) where J is 1, 2 and 3, we may show that ϕ_3^+ is zero at $x = -\infty$. ϕ_3^+ is generated by products of $\phi_1^{(r)}$, $\phi_1^{(s)}$ and $\phi_1^{(t)}$ where r, s, t are either 1, 2 or 3. More generally we will find that $\phi_m^+ = 0$ at $x = -\infty$ where m is an arbitrary positive number.

6. The oscillatory forces

We first consider an incident monochromatic wave. Let $\boldsymbol{f}_1$ denote the first-order oscillatory force. The x- and y-components of the force may be written

$$\boldsymbol{f}_1 \cdot \boldsymbol{i} = -K\mathrm{i}\int_0^{2\pi} \phi_1 \cos\theta\,\mathrm{d}\theta, \tag{103}$$

$$\boldsymbol{f}_1 \cdot \boldsymbol{j} = -K\mathrm{i}\int_0^{2\pi} \phi_1 \sin\theta\,\mathrm{d}\theta, \tag{104}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors along the x- and y-axes, respectively. Applying formula (55) for ϕ_1, (103) and (104) take the form

$$\boldsymbol{f}_1 \cdot \boldsymbol{i} = -\tfrac{1}{2}K\mathrm{i}\int_0^{2\pi} \sum_0^{\infty} A_{1m}\,\mathrm{e}^{\mathrm{i}m\theta}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta, \tag{105}$$

$$\boldsymbol{f}_1 \cdot \boldsymbol{j} = -\tfrac{1}{2}K\int_0^{2\pi} \sum_0^{\infty} A_{1m}\,\mathrm{e}^{\mathrm{i}m\theta}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta, \tag{106}$$

which gives

$$\boldsymbol{f}_1 \cdot \boldsymbol{i} = -K\pi\mathrm{i}A_{11}, \tag{107}$$

$$\boldsymbol{f}_1 \cdot \boldsymbol{j} = K\pi A_{11}. \tag{108}$$

Introducing the time variation, the formulae take the form

$$\boldsymbol{F}_1 \cdot \boldsymbol{i} \equiv \mathrm{Re}\,(\boldsymbol{f}_1 \cdot \boldsymbol{i}\,\mathrm{e}^{\mathrm{i}\tau}) = \mathrm{Re}\,(-\mathrm{i}\pi K A_{11}\,\mathrm{e}^{\mathrm{i}\tau}), \tag{109}$$

$$\boldsymbol{F}_1 \cdot \boldsymbol{j} \equiv \mathrm{Re}\,(\boldsymbol{f}_1 \cdot \boldsymbol{j}\,\mathrm{e}^{\mathrm{i}\tau}) = \mathrm{Re}\,(-\mathrm{i}\pi K A_{11}\,\mathrm{e}^{\mathrm{i}(\tau + \frac{1}{2}\pi)}). \tag{110}$$

Thus the x-component and the y-component of the first-order force have the same amplitude and a phase difference $\frac{1}{2}\pi$. This result was derived by Ogilvie (1963), using a different approach.

For the second-order oscillatory force, $\boldsymbol{f}_2$, we obtain from (26)

$$\boldsymbol{f}_2 \cdot \boldsymbol{i} = -2K\mathrm{i}\int_0^{2\pi} \phi_{22}\cos\theta\,\mathrm{d}\theta - \tfrac{1}{4}K\int_0^{2\pi} (\boldsymbol{\nabla}\phi_1)^2\cos\theta\,\mathrm{d}\theta, \tag{111}$$

$$\boldsymbol{f}_2 \cdot \boldsymbol{j} = -2K\mathrm{i}\int_0^{2\pi} \phi_{22}\sin\theta\,\mathrm{d}\theta - \tfrac{1}{4}K\int_0^{2\pi} (\boldsymbol{\nabla}\phi_1)^2\sin\theta\,\mathrm{d}\theta. \tag{112}$$

The last term in (11) may, using the boundary condition (19), be written

$$\tfrac{1}{8}K\int_0^{2\pi} \sum_{n=1,\,n'=1}^{\infty} A_{1n}A_{1n'}\,nn'\,\mathrm{e}^{\mathrm{i}(n+n')\theta}\,(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta, \tag{113}$$

which, by changing the order of summation and integration, is seen to be zero. Similarly we find that the last term in (112) is zero. Introducing (68) for ϕ_{22} we then obtain

$$f_2 \cdot i = -2i\pi K A_{21}, \tag{114}$$

$$f_2 \cdot j = 2\pi K A_{21}. \tag{115}$$

Hence the x- and y-components of the second-order oscillatory force also have the same amplitudes and a phase difference $\frac{1}{2}\pi$. The first part of this result, that the two amplitudes are equal, has been observed by Vada (1987) by numerical simulation.

The results may be extended to oscillations of order m with frequency $m\omega$. From the Bernoulli equation we find

$$p_{mm} = -mKi\phi_{mm} - \tfrac{1}{4}K(\nabla\phi)^2, \tag{116}$$

where p_{mm} denotes the dynamic pressure. ϕ is composed of $\phi_1, \phi_{22}, \ldots, \phi_{m-1,m-1}$ such that the product $\nabla\phi \cdot \phi$ is of order m. Similarly as for f_2, we show that the last term in (118) becomes zero, and we obtain

$$f_m \cdot i = -miK\pi A_{m1}, \tag{117}$$

$$f_m \cdot j = mK\pi A_{m1}. \tag{118}$$

Hence the x-component and the y-component of the oscillatory force of order m and frequency $m\omega$ have the same amplitudes and a phase differences $\frac{1}{2}\pi$.

A corresponding result is also true for incident bichromatic waves. The Bernoulli equation may then be written

$$p_2^+ = -\frac{\omega_1+\omega_2}{\omega}Ki\phi_2^+ - \tfrac{1}{4}K(\nabla\phi_1)^2, \tag{119}$$

where p_2^+ denotes dynamic pressure and $\phi_1 = \phi_1^{(1)} + \phi_2^{(2)}$, defined in §5. Exactly as for monochromatic waves it follows that the last term gives no contribution and we obtain from (101)

$$f_2^+ \cdot i = -\frac{\omega_1+\omega_2}{\omega} iK\pi A_{21}^+, \tag{120}$$

$$f_2^+ \cdot j = \frac{\omega_1+\omega_2}{\omega} K\pi A_{21}^+. \tag{121}$$

Also in this case we obtain that the x- and y-components of the oscillatory force have the same amplitudes and a phase difference $\frac{1}{2}\pi$. This result is consistent with numerical simulations by Friis, Grue & Palm (1991) who find, within the accuracy of the code, that the amplitudes are equal. The procedure may be extended to higher orders.

7. Summary and discussion

This paper discusses the nonlinear wave reflection of an incident monochromatic wave, caused by a circular cylinder submerged in a fluid layer of infinite depth. A reflected Fourier mode with frequency $m\omega$, m an arbitrary integer, will have components of order $m, m+2, m+4$, etc. It is shown in the paper that the component of order m, which is the dominant part of the mode, is not reflected. Hence for $m = 1$, for instance, the first-order reflection coefficient is zero, but probably not the third- or the fifth-order reflection coefficients. These are, however, usually very small.

Also incident bichromatic waves (and briefly multichromatic waves) are studied and it is shown that the second-order wave with frequency equal to the sum of two incident frequencies has no reflection. This is most likely not true for the second-order wave with difference-frequency.

It is also shown that the x-component and y-component of the oscillatory force of order m and frequency $m\omega$ have identical amplitudes and a phase difference $\frac{1}{2}\pi$. This result is also true for the second-order force (and higher-order forces) with frequency being the sum of the frequencies of two incident bichromatic waves.

It should be noted that the derived results are valid for arbitrary submergence of the body and arbitrary incident frequencies.

Our results for $m = 2$ and 3 are confirmed by laboratory experiments by Chaplin (1984). Thus he observes that the reflection of the 2- and 3-harmonic modes is negligible if the Keulegan–Carpenter number is unity or smaller. For these two harmonic modes he also finds that the x-component and y-component of the oscillatory force have the same amplitudes. He does not comment on higher harmonic modes, most likely because they are very small in his experiments. Among the practical applications of this theoretical investigation are the evaluation of the forces on long underwater tube bridges, which it is proposed to construct across Norwegian fjords and straits.

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Appendix

We consider the integral

$$\int_{-\infty}^{+\infty} G_2(x, y, x', 0) f_2(x')\, \mathrm{d}x'. \tag{A 1}$$

For $y' = , \log r/r^* = 0$. Hence G_2 is replaced by G_2', where $G_2'(x, y, x', y', K) = G'(x, y, x', y', 4K)$. Here G' is defined by (33). From (20) $f_2(x)$ consists of products of ϕ_1 and its x-derivatives. We notice from (61) that $\phi_1(x)$ is a function of x only through terms of the type $\exp(-\mathrm{i}k_1 x)$ where k_1 is positive. $f_2(x)$ is then a function of x through terms of the type $\exp(-\mathrm{i}(k_1+k_2)x)$ where k_1 and k_2 both are positive. Introducing now for G_2 the first integral on the right-hand side of (33) (with K replaced by $4K$) and integrating over x, we obtain that this part of G_2 gives a contribution proportional to $\delta(-k-k_1-k_2)$ where k, k_1 and k_2 all are positive. δ denotes the Dirac δ-function. Hence this contribution is zero and G_2 in (A 1) may be replaced by the last integral in (33). Changing the order of integration (A 1) may be written in the simpler form

$$\int_0^{\infty} \frac{\tilde{f}_2(k)\, \mathrm{e}^{-\mathrm{i}kx}}{k-4K}\, \mathrm{d}k, \tag{A 2}$$

where

$$\tilde{f}_2(k) = \int_{-\infty}^{\infty} f_2(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x. \tag{A 3}$$

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